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2002 J. Phys. A: Math. Gen. 35 7613

(http://iopscience.iop.org/0305-4470/35/35/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 7613–7628 PII: S0305-4470(02)38305-7

Intertwining relations for the matrix Calogero-like models: supersymmetry and shape invariance

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Received 18 June 2002
Published 22 August 2002
Online at stacks.iop.org/JPhysA/35/7613

Abstract

We investigate the intertwining relations for N-particle Calogero-like models with internal degrees of freedom. Starting from the well-known Dunkl–Polychronakos operators, we construct a new type of local (without exchange operation) differential operator. These operators intertwine the matrix Hamiltonians corresponding to irreducible representations of the permutation group S_N. In particular cases, this method allows us to construct a new class of exactly solvable Dirac-like equations and a new class of matrix models with shape invariance. We establish a connection with the approach of multi-dimensional supersymmetric quantum mechanics.

PACS numbers: 02.10.Yn, 02.30.Tb, 03.65.Ge, 11.30.Pb

1. Introduction

The exactly solvable quantum N-body problems have provided useful tools to investigate both formal algebraic and analytic properties with applications to different branches of physics. The most intensively studied models are the Calogero model (the many-body extension of the one-dimensional singular harmonic oscillator) [1, 2] and its various generalizations, the so-called Calogero-like models. The latter have either scalar [3–5] or matrix (with internal degrees of freedom) [6–9] natures. The Calogero-like models have been widely developed incorporating many-body forces [10], different root systems [11, 12] and multi-dimensions [13]. The supersymmetric extensions of Calogero-like models [14–18] also seem to be very promising.

In the papers [6, 7, 11, 19–22] different types of Dunkl operators were used for the investigation of Calogero-like models[1]. These operators intertwine Calogero-like

[1] We call the one-particle operators constructed in [6, 20] the Dunkl–Polychronakos (DP) operators to distinguish them from the genuine Dunkl operators set forth in the papers [7, 19, 21, 22], which are slightly different.

Hamiltonians and therefore allow us to construct the integrals of motion and the eigenfunctions (if they exist) for these models.

The characteristic trait of DP operators and of the corresponding integrals of motion derived from them is that they involve the coordinate exchange operators, and thus are nonlocal[2]. These techniques are briefly outlined in subsection 2.1.

On the other hand, the multi-dimensional supersymmetric quantum mechanics (SUSY QM) [23], applied to the Calogero-like models [14–17, 24] provides us with another set of the intertwining relations, where the matrix Calogero-like Hamiltonians of a specific type [18], are intertwined by the the supercharge operators. In this approach, both the Hamiltonians and the supercharge operators are local. The intertwining relations are the most important part of the SUSY QM algebra, which is clear from a number of generalizations of the standard SUSY QM, see, for example, [25, 26].

In the rest of section 2 these two approaches are unified; from the DP operators set forth in [6, 20] we construct new local operators of the first order in derivatives, which play the role of intertwining operators between the matrix Calogero-like Hamiltonians. For the Calogero–Sutherland model [3] this leads to a new class of exactly solvable Dirac-like (matrix and of the first order in derivatives) Hamiltonians. For the Calogero model with oscillator terms, the new intertwining relations give a new implementation of the shape invariance condition [15, 26–28].

In section 3, we consider the particular case of three-particle Calogero-like model, which is the simplest nontrivial realization of the method introduced in section 2. For models without oscillator terms (OT) the above-mentioned Dirac-like Hamiltonians coincide with the conventional Dirac Hamiltonians for a massless particle in a magnetic field. In this case, the 2×2 matrix Calogero-like Hamiltonians can be interpreted as Pauli Hamiltonians for the same system.

In section 4, we consider the class of Calogero-like matrix Hamiltonians described in [18]. For such models, the intertwining relations derived in section 2 are reduced to the well-known SUSY QM relations [23]. However, there is a wide class of models for which the intertwining relations introduced in section 2 are not reduced to any previously known ones. Clearly, SUSY QM is valid not only for the Calogero-like models, but for many other multi-dimensional and multiparticle models [23]. The question is how far the generalization of SUSY QM constructed below can be extended[3] to non Calogero-like models, and this deserves further attention.

A possible way to extend the formalism presented below is to consider the generalizations of the Calogero-like models incorporating many-body forces [10] and different root systems [12], for which the Dunkl operators also exist [11, 12]. Therefore, one can construct from these local intertwining operators analogous to those of the present paper.

2. Intertwining operators of first order in derivatives

2.1. Dunkl–Polychronakos operators [6, 20] for Calogero-like models

Let us consider a one-dimensional quantum system of N particles with coordinates x_i. Let M_{ij} be the operator that exchanges the coordinates of the ith and jth particles[4]. The DP operators

[2] By the words 'nonlocal' and 'local' we mean 'containing exchange operators' and 'containing no exchange operators'.
[3] One example of such an extension (for the three-particle case) is given in footnote 15.
[4] Throughout the paper, the lower case Roman indices range from 1 to N.

are defined [6, 20] as

$$\pi_i = -\mathrm{i}\partial_i + \mathrm{i}\sum_{j\neq i} V_{ij} M_{ij} = \pi_i^\dagger \qquad V_{ij} \equiv V(x_i - x_j)$$
$$\partial_i \equiv \frac{\partial}{\partial x_i} \qquad V(x) = V(-x) = V^*(x). \tag{1}$$

The operators π_i are one-particle operators, i.e.

$$M_{ij}\pi_i = \pi_j M_{ij} \qquad [M_{ij}, \pi_k] = 0 \qquad k \neq i, j. \tag{2}$$

Their commutators can be written as

$$[\pi_i, \pi_j] = \sum_{k\neq i,j} V_{ijk}[M_{ijk} - M_{jik}] \qquad \text{where} \quad V_{ijk} \equiv V_{ij}V_{jk} + V_{jk}V_{ki} + V_{ki}V_{ij} \tag{3}$$

and M_{ijk} are the operators of cyclic permutations in three indices:

$$M_{ijk} \equiv M_{ij}M_{jk} = M_{jki} = M_{kij} = M_{jik}^\dagger.$$

For both the Calogero–Sutherland (CS) models

$$V(x) = l\cot x \qquad \text{(trigonometric or TCS model)}$$
$$V(x) = l\coth \qquad \text{(hyperbolic)} \tag{4}$$

and the delta-function model

$$V(x) = l\,\mathrm{sign}\,x \tag{5}$$

the function $V_{ijk} = l^2$, so that

$$[\pi_i, \pi_j] = l^2 \sum_{k\neq i,j}[M_{ijk} - M_{jik}]. \tag{6}$$

The Hamiltonians for these models are[5]

$$H = -\Delta + \sum_{i\neq j}\left[V'_{ij}M_{ij} + V_{ij}^2\right] = \sum_i \pi_i^2 + \frac{l^2}{3}\sum_{i\neq j\neq k\neq i} M_{ijk} \tag{7}$$

where $\Delta \equiv \sum_i \partial_i\partial_i$ and $V'_{ij} \equiv V'(x_i - x_j)$. It is known [6, 20] that in this case

$$[\pi_i, H] = 0. \tag{8}$$

In the case of the Calogero model

$$V(x) = l/x \qquad V_{ijk} = 0 \qquad [\pi_i, \pi_j] = 0 \tag{9}$$

and the equations (7) and (8) remain valid. What is more, the DP operators themselves mutually commute. However, this model does not have a discrete spectrum.

The Calogero model is usually considered in a harmonic confining potential (we abbreviate this to Calogero oscillator (CO)). For this model, the following operators should be introduced [6, 20], (see also [7, 11, 21, 22]):

$$a_i^\pm = \pi_i \pm \imath\omega x_i \qquad \left(a_i^+\right)^\dagger = a_i^-. \tag{10}$$

The Hamiltonian can be written as

$$H_{\mathrm{CO}} = \sum_i a_i^+ a_i^- + l\omega\sum_{i\neq j} M_{ij} = -\Delta + \omega^2\sum_i x_i^2 + \sum_{i\neq j}\frac{l(l - M_{ij})}{(x_i - x_j)^2} + N\omega. \tag{11}$$

It has been proven [6] that the operators H_{CO} and $a_i^\pm$ form the oscillator algebra:

$$\left[H_{\mathrm{CO}}, a_j^\pm\right] = \pm 2\omega a_j^\pm. \tag{12}$$

[5] The following procedure is applicable to an arbitrary set of operators π_i, provided they satisfy equations (2) and (6) and that the Hamiltonian is given by the second equality of equation (7).

2.2. The local form of the Hamiltonians

Let us consider an irreducible representation A of the permutation group S_N realized on real vector functions $f_\alpha(x_1, \ldots, x_N)$; $\alpha = 1, \ldots, \dim A$ by the matrices $\mathbf{T}^A_{ij}$

$$M_{ij} f_\alpha = \left(T^A_{ij}\right)_{\beta\alpha} f_\beta \tag{13}$$

where $\left(T^A_{ij}\right)_{\beta\alpha} = \left(T^A_{ij}\right)_{\alpha\beta}$ are the (constant) matrix elements of the permutation operator M_{ij} in the representation A. Below, we assume summation over the repeated indices, unless specified otherwise. We also use the fact [30] that $\mathbf{T}^A_{ij}$ are real symmetric orthogonal matrices.

It is useful to introduce the vector notations

$$\mathbf{f} = \mathbf{e}_\alpha f_\alpha \tag{14}$$

where the constant vectors $\mathbf{e}_\alpha$ ($\alpha = 1, \ldots, \dim A$) form a basis in the space of the representation A. Then it is also helpful to define the operator $\mathbf{T}^A_{ij}$ in the vector form:

$$\mathbf{T}^A_{ij}\mathbf{e}_\alpha = \mathbf{e}_\beta(\mathbf{e}_\beta)^\dagger \mathbf{T}^A_{ij}\mathbf{e}_\alpha \equiv \left(T^A_{ij}\right)_{\alpha\beta} \mathbf{e}_\beta. \tag{15}$$

Multiplying equation (13) onto $\mathbf{e}_\alpha$ and using equation (15), we obtain

$$M_{ij}\mathbf{f} = \mathbf{T}^A_{ij}\mathbf{f} \tag{16}$$

where M_{ij} act only on the arguments of f_α, and $\mathbf{T}^A_{ij}$ only on the vectors $\mathbf{e}_\alpha$.

All the Hamiltonians H from the previous subsection can be written as $H = H_{\mathrm{scal}} + V'_{ij} M_{ij}$, where H_{scal} are scalar operators containing no exchange operator terms:

$$H_{\mathrm{scal}} = -\Delta + \sum_{i\neq j} V^2_{ij} \tag{17}$$

for the models without OT (4)–(5), and

$$H_{\mathrm{scal}} = -\Delta + \omega^2 \sum_i x_i^2 + \sum_{i\neq j} \frac{l^2}{(x_i - x_j)^2} + N\omega \tag{18}$$

for the CO model (11). The Hamiltonians H act on the functions[6] from the representation A as

$$H f_\alpha = H^A_{\beta\alpha} f_\beta \qquad H^A_{\beta\alpha} \equiv H_{\mathrm{scal}}\delta_{\alpha\beta} + \sum_{i\neq j} V'_{ij} \left(T^A_{ij}\right)_{\beta\alpha} = H^A_{\alpha\beta} \tag{19}$$

or, in the vector form,

$$H\mathbf{f} = \mathbf{H}^A\mathbf{f}.$$

Note that, if $\mathbf{f}$ satisfies equation (16), then $H\mathbf{f}$ satisfies it too, since equation (16) is equivalent to the condition: $\mathbf{T}^A_{ij} M_{ij}\mathbf{f} = \mathbf{f}$ (no summation over i, j). The latter condition is satisfied for $H\mathbf{f}$, because $\left[H, \mathbf{T}^A_{ij} M_{ij}\right] = 0$.

2.3. The intertwining operators in the local form

The matrix Hamiltonians (19) do not contain exchange operators M_{ij} explicitly. Our aim now is to remove M_{ij} from the DP operators (1) and (10), and to rewrite equations (8) and (12) in terms of local operators only.

Let us study the action of the DP operators on the symmetric functions satisfying equations (13) and (16). The expression $\pi_i f_\alpha$ no longer satisfies equation (13) even when

[6] Note that the functions $\mathbf{f}$ are not necessarily eigenfunctions of H. In this paper, we do not discuss the symmetry properties of the eigenfunctions of the Calogero-like models.

f_α satisfies it. Instead, $\pi_i f_\alpha$ transforms under the action of M_{ij} as an object from the direct product of representations for π_i and f_α. Of course, the DP operators transform under S_N in accordance with equation (2). However, π_i belong to a reducible representation of S_N because $\pi_1 + \cdots + \pi_N$ realizes the absolutely symmetric representation. Therefore, it is helpful to go to the well-known Jacobi coordinates [29]

$$\begin{aligned} y_\xi &= \frac{1}{\sqrt{\xi(\xi+1)}}(x_1 + \cdots + x_\xi - \xi x_{\xi+1}) \qquad 1 \leqslant \xi \leqslant N-1 \\ y_N &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N} x_i \end{aligned} \tag{20}$$

or[7] $y_k = R_{km}x_m$, where the orthogonal matrix R is determined by equation (20). The derivatives are connected by the same matrix, $\partial/\partial y_k = R_{km}\partial/\partial x_m$, because R is an orthogonal matrix. Similarly, we can write the DP operators in the Jacobi variables

$$\begin{aligned} \rho_\xi &= \frac{1}{\sqrt{\xi(\xi+1)}}(\pi_1 + \cdots + \pi_\xi - \xi\pi_{\xi+1}) \qquad 1 \leqslant \xi \leqslant N-1 \\ \rho_N &= \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\pi_j = -\frac{\mathrm{i}}{\sqrt{N}}(\partial_1 + \cdots + \partial_N) \end{aligned}$$

or $\rho_k = R_{km}\pi_m$. The operators ρ_ξ now transform under S_N as the irreducible representation Γ with the Young tableau[8] $(N-1, 1)$. Similar to equation (13), this fact can be written as

$$M_{ij}\rho_\xi = \left(T_{ij}^{\Gamma}\right)_{\lambda\xi}\rho_\lambda. \tag{21}$$

This is a property of the Jacobi variables (see, for example, [18]).

The object $\rho_\xi f_\alpha$ transforms under the action of S_N as the interior product $\Gamma \times A$ of the representations A and Γ, or, in more detail, in accordance with the formulae (13) and (21), as

$$M_{ij}\rho_\xi f_\alpha = \left(T_{ij}^{\Gamma}\right)_{\lambda\xi}\left(T_{ij}^{A}\right)_{\beta\alpha}\rho_\lambda f_\beta.$$

As outlined in the book [30] (chapter 7, section 13), the interior product $\Gamma \times A$ contains only the irreducible representations of S_N, whose Young tableaux differ from the tableau for A by no more than the position of one cell (but not necessarily all of them). For example, for the absolutely symmetric representation of S_N with the Young tableau (N), obviously,

$$\Gamma \times (N) = \Gamma$$

and the result does not contain (N).

Let B be some irreducible representation that appears in $\Gamma \times A$. Then we can extract its contribution to $\Gamma \times A$ with the help of the Clebsch–Gordan coefficients $(\Gamma\xi, A\alpha|B\sigma) \equiv (\xi\alpha|\sigma)$:

$$g_\sigma = (\xi\alpha|\sigma)\rho_\xi f_\alpha = D_{\sigma\alpha} f_\alpha \qquad D_{\sigma\alpha} \equiv (\xi\alpha|\sigma)\rho_\xi. \tag{22}$$

The resulting function g_σ satisfies the analogue of equation (13) for the representation B:

$$M_{ij}g_\sigma = \left(T_{ij}^{B}\right)_{\delta\sigma} g_\delta. \tag{23}$$

This can be checked directly by substituting equation (22) into equation (23) and making use of the following expression[9]

[7] The indices of the Jacobi variables denoted by Greek letters range from 1 to $N-1$; those denoted by Roman letters range from 1 to N.

[8] The standard notation [30] for a Young diagram containing λ_i cells in the ith line is $(\lambda_1, \ldots, \lambda_n)$; if the diagram contains m identical lines with μ cells, it is denoted by $(\ldots, \mu^m, \ldots)$.

[9] The expression (24) (see [30], formula (5.114)) is actually a necessary and sufficient condition for $(C\xi, A\alpha|B\sigma)$ to be a Clebsch–Gordan coefficient for arbitrary representations A, B and $C \in A \times B$.

$$\left(T_{ij}^{\Gamma}\right)_{\lambda\xi}\left(T_{ij}^{A}\right)_{\beta\alpha}(\xi\alpha|\sigma)=\left(T_{ij}^{B}\right)_{\delta\sigma}(\lambda\beta|\delta) \tag{24}$$

and the fact that $\mathbf{T}_{ij}$ are Hermitian matrices.

On the functions that satisfy equation (13), the operator $D_{\sigma\alpha}$ acts as

$$D_{\sigma\alpha}f_{\alpha}=D_{\sigma\alpha}^{A}f_{\alpha}\qquad D_{\sigma\alpha}^{A}=(\xi\beta|\sigma)\left(\rho_{\xi}^{A}\right)_{\beta\alpha}=(\xi\beta|\sigma)R_{\xi k}\left(\pi_{k}^{A}\right)_{\beta\alpha} \tag{25}$$

$$\left(\pi_{k}^{A}\right)_{\beta\alpha}=-\mathrm{i}\partial_{k}\delta_{\beta\alpha}+\mathrm{i}\sum_{m\neq k}V_{km}\left(T_{km}^{A}\right)_{\beta\alpha}. \tag{26}$$

For models without OT $[H,\pi_i]=0$, so $[H,D_{\sigma\alpha}]=0$. Hence, for any f_α

$$HD_{\sigma\alpha}f_{\alpha}=D_{\sigma\alpha}Hf_{\alpha}. \tag{27}$$

For all f_α which satisfy equation (13), $H_{\beta\alpha}^{A}f_{\beta}$ satisfies equation (13) (see the end of the previous subsection), and $D_{\sigma\alpha}f_\alpha$ satisfies equation (23). Using these symmetry properties, we can expand the sides of the equation (27) as

$$(\mathrm{lhs})=H_{\delta\sigma}^{B}D_{\delta\alpha}f_{\alpha}=H_{\delta\sigma}^{B}D_{\delta\alpha}^{A}f_{\alpha}\qquad(\mathrm{rhs})=D_{\sigma\alpha}H_{\beta\alpha}^{A}f_{\beta}=D_{\sigma\alpha}^{A}H_{\beta\alpha}^{A}f_{\beta}.$$

Taking into account that $\mathbf{H}^A$, $\mathbf{H}^B$ are symmetric matrices in the internal indices, we can see that for the functions satisfying equation (13)

$$H_{\sigma\delta}^{B}D_{\delta\beta}^{A}=D_{\sigma\alpha}^{A}H_{\alpha\beta}^{A}. \tag{28}$$

Similar to the above, for the CO model from equation (12) it follows that $HD_{\sigma\alpha}^{\pm}=D_{\sigma\alpha}^{\pm}(H\pm\omega)$. Following the same route that led us from equation (27) to (28), we can conclude that for functions satisfying equation (13)

$$H_{\delta\sigma}^{B}D_{\delta\beta}^{A\pm}=D_{\sigma\alpha}^{A\pm}\left(H_{\alpha\beta}^{A}\pm2\omega\delta_{\alpha\beta}\right) \tag{29}$$

where

$$D_{\sigma\alpha}^{A\pm}=(\xi\beta|\sigma)R_{\xi j}\left[\left(\pi_{j}^{A}\right)_{\beta\alpha}\pm\mathrm{i}\omega x_{j}\delta_{\alpha\beta}\right]\qquad\left(\pi_{j}^{A}\right)_{\beta\alpha}=-\mathrm{i}\partial_{j}\delta_{\beta\alpha}+\mathrm{i}l\sum_{m\neq j}\frac{\left(T_{jm}^{A}\right)_{\beta\alpha}}{x_{j}-x_{m}}. \tag{30}$$

Note that all terms in equations (28) and (29) are local; they contain no exchange operators M_{ij}.

2.4. The operatorial nature of the intertwining relations

Equations (28) and (29) are not yet operatorial intertwining relations such as, for example, the SUSY QM ones (see [23, 27, 31]), because the former are valid only for functions that satisfy the symmetry condition (13). For functions outside that class equations (28) and (29), generally speaking, may no longer be satisfied.

However, we can prove that they are satisfied on all functions and thus are operatorial intertwining relations, by making use of the following.

Theorem 1. *Let A be some representation of S_N. Let $L_{\alpha\beta}$ be some linear differential operator of finite order with the coefficients being rational matrix functions of the variables x_i, or $\sin x_i, \cos x_i$, or $\mathrm{sh}x_i, \mathrm{ch}x_i$ (but no two of them simultaneously), singular at*[10] *$U=\{\mathbf{x}|\exists i,j:i\neq j, x_i=x_j\}$ at most. The coefficients are matrices of dimension*

[10] By $\mathbf{x}$ without an index, we mean the vector $(x_1,\ldots,x_N)$ with N elements.

$\dim A \times \dim A$. *Then, if*

$$L_{\alpha\beta} f_\beta = 0$$

for all f_β *satisfying equation (13), then* $L \equiv 0$ *as an operator.*

The proof of this theorem can be found in appendix A. Using theorem 1 for the difference between the left-hand and right-hand sides of equations (28) and (29) we can conclude that the latter are satisfied operatorially[11].

In particular, this means that, when the initial representation A coincides with the resulting representation B, the operators $\mathbf{D}^A$ are integrals of motion for the (trigonometric and hyperbolic) matrix CS models. Therefore, each CS matrix model, corresponding to a representation A such that $A \in \Gamma \times A$, has a local integral of motion $\mathbf{D}^A$ of the first order in derivatives. An example of a model from this class is given in section 3. Note that, for example, the models with $A = (N)$ or $A = (1^N)$ lie outside this class.

In the case of the TCS system, $\mathbf{H}^A$ are exactly solvable (see, for example, [8]) Hamiltonians with a discrete spectrum and a finite dimensional degeneracy of levels. The fact that equations (28) and (29) are satisfied operatorially when $B = A$ allows us to find also the spectrum and the eigenfunctions of $\mathbf{D}^A$, i.e. the normalizable functions $f_\alpha(\mathbf{x}), \alpha = 1, \ldots, \dim A$:

$$D^A_{\alpha\beta} f_\beta = \epsilon f_\alpha.$$

The operators $\mathbf{D}^A$ for the TCS system may be considered here as Dirac-like Hamiltonians of first order in derivatives:

$$D^A_{\sigma\alpha} = (\xi\beta|\sigma) R_{\xi j} \left[-\mathrm{i}\partial_j \delta_{\beta\alpha} + \mathrm{i}l \sum_{m\neq j} \cot(x_j - x_m) \left(T^A_{jm}\right)_{\beta\alpha} \right].$$

From the commutation relations (28) it follows that for a given A, the operators $\mathbf{H}^A$ and $\mathbf{D}^A$ can be diagonalized simultaneously[12]. In more detail, let $\mathbf{f}_b^{(n)}$ be degenerate eigenstates of $\mathbf{H}^A$ with energy E_n

$$\mathbf{H}^A \mathbf{f}_b^{(n)} = E_n \mathbf{f}_b^{(n)} \tag{31}$$

and $b = 1, \ldots, J$ (J is the degree of degeneracy). Then after acting by $\mathbf{D}^A$ on both sides of the equality (31) we see that $\mathbf{D}^A \mathbf{f}_b^{(n)}$ also satisfies equation (31). Hence[13], there exists a constant $J \times J$ matrix[14] $F^{(n)}$:

$$\mathbf{D}^A \mathbf{f}_b^{(n)} = F^{(n)}_{bc} \mathbf{f}_c^{(n)}.$$

Because $\mathbf{H}^A$ is Hermitian, we can choose $\mathbf{f}_b^{(n)}$ that constitute a basis of vector functions with $\dim A$ components on $\Re^N$. Because $\mathbf{D}^A$ is Hermitian, one can check that $F^{(n)}$ is also Hermitian, and it can be diagonalized by a unitary rotation

$$U^{(n)}_{ab} F^{(n)}_{bc} U^{(n)*}_{dc} = \epsilon^{(n)}_a \delta_{ac}.$$

The functions

$$\mathbf{g}_a^{(n)}(\mathbf{x}) = U^{(n)}_{ab} \mathbf{f}_b^{(n)}(\mathbf{x})$$

[11] Theorem 1 cannot be applied to the model with $V(x) = l \operatorname{sign} x$, because equation (28) may then contain delta-function-like singularities at U.

[12] Compare with the SUSY QM intertwining relations which were used to find a part of the spectrum of Hamiltonians in one [32, 33] or two [26] dimensions.

[13] We assume for simplicity that the action of $\mathbf{D}^A$ does not destroy the normalizablilty of $\mathbf{f}_b^{(n)}$.

[14] Note that the matrix elements $F^{(n)}_{bc}$ are *not* $\dim A \times \dim A$ matrices, but just scalar constants. In other words, they do not affect the vector structure of $\mathbf{f}_b^{(n)}$.

are eigenfunctions of the operator $\mathbf{D}^A$ with energies $\epsilon_a^{(n)}$. Because $U_{ab}^{(n)}$ are unitary matrices, the functions $\mathbf{g}_a^{(n)}$ constitute a basis of the vector functions on $\Re^N$. Hence, they form a full set of eigenfunctions of $\mathbf{D}^A$.

For the CO model, the case of $B = A$ is also interesting. Equation (29) will then take the following form

$$H_{\delta\sigma}^{A} D_{\delta\beta}^{A\pm} = D_{\sigma\alpha}^{A\pm}\left(H_{\alpha\beta}^{A} \pm 2\omega\delta_{\alpha\beta}\right) \tag{32}$$

where $\mathbf{D}^{A\pm}$ are still defined by equation (30), and will be satisfied operatorially. They are the relations of the oscillator-like algebra (similar to equation (12)). In the language of SUSY QM, this means that each CO matrix model for a representation $A \in \Gamma \times A$ obeys shape invariance (SI) in $N-1$ dimensions (because the centre-of-mass motion is decoupled). Let us stress that this is the first example of SI in several dimensions realized by local operators of the first order in derivatives (see the attempts in [15]). The nonlocal SI of the Calogero model (see equation (12)) has been described in [6, 7, 11, 20–22]; a model with a two-dimensional SI of the second order in derivatives has been proposed in [26].

3. Examples with $N = 3$

We assume that $N = 3$, and both representations A and B are equal to $\Gamma = (2, 1)$. Then the functions that satisfy equation (16) are

$$\mathbf{f} = \begin{pmatrix} f_{211} \\ f_{121} \end{pmatrix}$$

where the index $\alpha = (211), (121)$ enumerates the partitions of the Young tableau (2, 1) (see [30] or [34], the end of chapter 1):

$$\begin{aligned} &P^{+} f_{211} = P^{-} f_{211} = 0 \qquad && M_{12} f_{211} = f_{211} \\ &P^{+} f_{121} = P^{-} f_{121} = 0 \qquad && M_{12} f_{121} = -f_{121} \\ &P^{\pm} = 1 + M_{12}M_{13} + M_{13}M_{12} \pm (M_{12} + M_{23} + M_{13}). && \end{aligned}$$

$P^{\pm}$ are the projectors onto the symmetric/antisymmetric representations of S_3. The matrices of the representation Γ for $N = 3$ have the form

$$\mathbf{T}_{12}^{\Gamma} = \sigma_3 \qquad \mathbf{T}_{23}^{\Gamma} = \frac{\sqrt{3}}{2}\sigma_1 - \frac{1}{2}\sigma_3 \qquad \mathbf{T}_{31}^{\Gamma} = -\frac{\sqrt{3}}{2}\sigma_1 - \frac{1}{2}\sigma_3 \tag{33}$$

where σ_i are the Pauli matrices. The Hamiltonians for the Calogero-like systems without OT for the representation Γ have the form (19):

$$\mathbf{H}^{\Gamma} = -\Delta + \sum_{i\neq j}\left[V_{ij}^{2} + V_{ij}'\mathbf{T}_{ij}^{\Gamma}\right]. \tag{34}$$

As for the DP operators in the Jacobi variables, one can check that ρ_1 has the same symmetry as f_{121}, and ρ_2 the same as f_{211}. The Clebsch–Gordan coefficients for $\Gamma \times \Gamma \to \Gamma$ are written in the end of chapter 7 of [30]. Inserting these and $\mathbf{T}_{ij}^{\Gamma}$ from equation (33) into the definition of $\mathbf{D}^{\Gamma}$ (expressions (25) and (26)), we can conclude after some algebra that for the Calogero-like models without OT

$$\mathbf{D}^{\Gamma} = \frac{1}{\sqrt{2}}\left[-\mathrm{i}\sigma_3\frac{\partial}{\partial y_2} + \mathrm{i}\sigma_1\frac{\partial}{\partial y_1} - \sqrt{2}\sigma_2(V_{12} + V_{23} + V_{31})\right] \tag{35}$$

where y_ξ are the Jacobi variables (20). Equation (28) for $\mathbf{H}^{\Gamma}$ from equation (34) and $\mathbf{D}^{\Gamma}$ from equation (35) is satisfied operatorially, i.e. $[\mathbf{H}^{\Gamma}, \mathbf{D}^{\Gamma}] = 0$. In fact, a stronger statement for

$N = 3$ can be proven by direct calculation

$$(\mathbf{D}^{\Gamma})^2 = \frac{1}{2}\left[\mathbf{H}^{\Gamma} + \frac{\partial^2}{\partial y_3^2}\right] + C \tag{36}$$

where C is a real constant. What is more[15], it is true operatorially even for the case $V(x) = l\,\mathrm{sign}\,x$. In this case, we have been unable to prove equation (28) for arbitrary representation A, but for $N = 3$ and $A = \Gamma$ it turns out to be true.

Equation (36) signifies that the operator $\mathbf{D}^{\Gamma}$ realizes a sort[16] of a 'square root' of $\mathbf{H}^{\Gamma}$. This, in particular, means that the spectrum and eigenfunctions of $\mathbf{D}^{\Gamma}$ itself can be found more easily than in the general case described in section 2, if the spectrum and eigenfunctions of $\mathbf{H}^{\Gamma}$ are known.

For the TCS model, the operator $\mathbf{D}^{\Gamma}$ (35) has the following form

$$\mathbf{D}^{\Gamma} = \frac{1}{\sqrt{2}}\left[-\mathrm{i}\sigma_3\frac{\partial}{\partial y_2} + \mathrm{i}\sigma_1\frac{\partial}{\partial y_1} - l\sqrt{2}\sigma_2\left(\cot(\sqrt{2}y_1) + \cot\left(-\frac{\sqrt{2}}{2}y_1 + \sqrt{\frac{3}{2}}y_2\right)\right.\right.$$
$$\left.\left. + \cot\left(-\frac{\sqrt{2}}{2}y_1 - \sqrt{\frac{3}{2}}y_2\right)\right)\right].$$

The eigenfunctions of this operator are two-component column functions $\mathbf{g}(y_1, y_2)$:

$$\mathbf{D}^{\Gamma}\mathbf{g}(y_1, y_2) = \epsilon\mathbf{g}(y_1, y_2). \tag{37}$$

In this case, it follows from equation (36) that $\mathbf{g}$ is also an eigenfunction of $\mathbf{H}^{\Gamma}$ with energy $E = 2(\epsilon^2 - C)$.

Let us prove now that all the eigenfunctions of $\mathbf{D}^{\Gamma}$ can be obtained from those of $\mathbf{H}^{\Gamma}$. Let $\mathbf{f}(\mathbf{x})$ be an eigenfunction of $\mathbf{H}^{\Gamma}$ with energy E: $\mathbf{H}^{\Gamma}\mathbf{f} = E\mathbf{f}$ and zero total momentum. Then the following alternatives should be considered.

(a) $\mathbf{f}$ itself is already an eigenfunction of $\mathbf{D}^{\Gamma}$, i.e. it satisfies equation (37). Then the corresponding eigenvalue is $\epsilon = \pm\sqrt{E/2 + C}$.

(b) $\mathbf{D}^{\Gamma}\mathbf{f} \equiv \mathbf{u}$ and $\mathbf{f}(\mathbf{x})$ are linearly independent. Then one can check that $\mathbf{D}^{\Gamma}\mathbf{u} = (C + E/2)\mathbf{f}$, and $\mathbf{u}(\mathbf{x})$ is also an eigenfunction of $\mathbf{H}^{\Gamma}$ with energy E. Thus, the eigenfunctions of $\mathbf{H}^{\Gamma}$, which are not those of $\mathbf{D}^{\Gamma}$, form pairs in which $\mathbf{D}^{\Gamma}$ transforms each member into another. If $\mathbf{f}$ is normalizable then $\mathbf{u}$ is too, because

$$\langle\mathbf{u}|\mathbf{u}\rangle = \langle\mathbf{f}|(\mathbf{D}^{\Gamma})^2|\mathbf{f}\rangle = (E/2 + C)\langle\mathbf{f}|\mathbf{f}\rangle < +\infty.$$

From each such pair one can construct two eigenfunctions of $\mathbf{D}^{\Gamma}$

$$\mathbf{f} \pm (C + E/2)^{-1/2}\mathbf{u}$$

with energies[17] $\epsilon = \pm\sqrt{C + E/2}$.

The Hamiltonian $\mathbf{H}^{\Gamma} + \frac{\partial^2}{\partial y_3^2}$ is Hermitian, so its eigenfunctions form a basis in the space of vector functions on $\Re^2$. Therefore, the eigenfunctions of $\mathbf{D}^{\Gamma}$ constructed above also form a basis; hence, they form a full set of eigenfunctions of $\mathbf{D}^{\Gamma}$.

[15] One could even replace the term $V_{12} + V_{23} + V_{31}$ in equation (35) by an arbitrary function $v(y_1, y_2)$. Then the square of $\mathbf{D}^{\Gamma}$ would remain the sum of the Laplacian and a momentum independent 2×2 matrix potential. However, we do not know any cases where this sum is an exactly solvable Hamiltonian, except for those given in this text.

[16] The operator (35) can be viewed as a Dirac operator for a massless fermion in three dimensions (y_1, y_2, y_3) in the magnetic field that does not depend on y_3 and is orthogonal to the axis y_3. The component of the fermion's momentum along the axis y_3 should be zero. The Hamiltonian (34) is then the Pauli Hamiltonian for the same system [31, 35].

[17] The situation $C + E/2 < 0$ is impossible because otherwise equation (37) would remain valid, but with imaginary ϵ. The operator $\mathbf{D}^{\Gamma}$ is Hermitian, so $C + E/2 \geqslant 0$.

For the CO model, the Hamiltonian (19) for the representation Γ has the form

$$\mathbf{H}^{\Gamma} = -\Delta + \omega^2 \sum_i x_i^2 + \sum_{i\neq j} \frac{l\big(l - \mathbf{T}_{ij}^{\Gamma}\big)}{(x_i - x_j)^2} + 3\omega$$

where $\mathbf{T}_{ij}^{\Gamma}$ are defined in equation (33). The intertwining operators (30) can be rewritten as

$$\mathbf{D}^{\Gamma\pm} = \frac{1}{\sqrt{2}} \left[-\mathrm{i}\sigma_3 \left(\frac{\partial}{\partial y_2} \mp \omega y_2 \right) + \mathrm{i}\sigma_1 \left(\frac{\partial}{\partial y_1} \mp \omega y_1 \right) - \sqrt{2}\sigma_2 \left(\frac{1}{y_1} + \frac{1}{-\frac{1}{2}y_1 + \frac{\sqrt{3}}{2}y_2} + \frac{1}{-\frac{1}{2}y_1 - \frac{\sqrt{3}}{2}y_2} \right) \right].$$

The operatorial relations (32) of the oscillator-like algebra

$$[\mathbf{H}^{\Gamma}, \mathbf{D}^{\Gamma\pm}] = \pm 2\omega \mathbf{D}^{\Gamma\pm}$$

correspond to the SI of the matrix 2×2 Hamiltonian $\mathbf{H}^{\Gamma}$ in two dimensions (because the centre-of-mass motion is decoupled).

4. Connection with ordinary SUSY QM

In this section, we restrict ourselves to the class of representations with Young tableaux of the form

$$A = (N - n, 1^n) \qquad n = 1, \ldots, N. \tag{38}$$

It was proven in [18] that for this class of representations we can choose a basis $\mathbf{e}_\alpha$ (see equation (14)) with the help of the fermionic creation/annihilation operators $\psi_i, \psi_i^+; i = 1, \ldots, N$:

$$\begin{aligned} &\{\psi_i, \psi_j\} = 0 && \left\{\psi_i^+, \psi_j^+\right\} = 0 && \left\{\psi_i, \psi_j^+\right\} = \delta_{ij} \\ &\psi_i|0\rangle = 0 && i, j = 1, \ldots, N && \langle 0|0\rangle = 1. \end{aligned} \tag{39}$$

It is useful to introduce also the fermionic analogues ϕ_k^+ of the Jacobi variables (20) (see [18])

$$\phi_k^+ = R_{km}\psi_m^+ \qquad \phi_k = R_{km}\psi_m$$

where R_{km} are defined in equation (20). The fermionic Jacobi variables obey anticommutation relations similar to equation (39)

$$\begin{aligned} &\{\phi_k, \phi_m\} = 0 && \left\{\phi_k^+, \phi_m^+\right\} = 0 && \left\{\phi_k, \phi_m^+\right\} = \delta_{km} \\ &\phi_k|0\rangle = 0 && k, m = 1, \ldots, N. && \end{aligned} \tag{40}$$

Now we can define the basis[18] $\mathbf{e}_\alpha$

$$\mathbf{e}_\alpha = \phi_{\alpha_1}^+ \ldots \phi_{\alpha_n}^+|0\rangle \equiv |\alpha_1 \ldots \alpha_n\rangle \qquad \alpha_i = 1, \ldots, N - 1 \tag{41}$$

where $\alpha \equiv (\alpha_1 \ldots \alpha_n)$ is a multi-index with values in the fermionic number space, and

$$(\mathbf{e}_\alpha)^\dagger \mathbf{e}_\alpha = \langle \alpha_n \ldots \alpha_1 | \alpha_1 \ldots \alpha_n \rangle = 1 \qquad (\mathbf{e}_\alpha)^\dagger = (|\alpha_1 \ldots \alpha_n\rangle)^\dagger = \langle 0|\phi_{a_n} \ldots \phi_{a_1} \equiv \langle \alpha_n \ldots \alpha_1|. \tag{42}$$

(No sumation over α is implied.)

[18] The fermionic operators ϕ_N, ϕ_N^+ do not enter into $\mathbf{e}_\alpha$ because they correspond to the centre-of-mass degree of freedom which is decoupled.

Because of equation (40), it is sufficient to include into the basis only the vectors $\mathbf{e}_\alpha$ with, say, $\alpha_1 < \cdots < \alpha_n$, and the summation over α will be done over such vectors only.

It has also been proven in [18] that, for the basis (41), the operator $\mathbf{T}^A_{ij}$ in the vector form (15) can be realized as

$$\left(T^A_{ij}\right)_{\alpha\beta}\mathbf{e}_\beta = \mathbf{T}^A_{ij}\mathbf{e}_\alpha = \mathbf{T}^A_{ij}|\alpha_1\ldots\alpha_n\rangle = K_{ij}|\alpha_1\ldots\alpha_n\rangle \tag{43}$$

where

$$\hat{K}_{ij} \equiv \psi^+_i\psi_j + \psi^+_j\psi_i - \psi^+_i\psi_i - \psi^+_j\psi_j + 1 = 1 - \left(\psi^+_i - \psi^+_j\right)(\psi_i - \psi_j) = \hat{K}_{ji} = (\hat{K}_{ij})^\dagger. \tag{44}$$

It follows from equation (43) that all the Hamiltonians $\mathbf{H}^A$ (19) with A from the class (38) take the same form in the basis (41):

$$H = H_{\mathrm{scal}} + \sum_{i\neq j} V'_{ij}K_{ij}. \tag{45}$$

Let us consider the intertwining relations (28) for the Calogero-like models without OT for A from the class (38). From equations (45) and (17) it follows that the Hamiltonians $\mathbf{H}^A$, $\mathbf{H}^B$ in equation (28) have the form

$$H = -\Delta + \sum_{i\neq j}\left[V'_{ij}K_{ij} + V^2_{ij}\right]. \tag{46}$$

One may notice that this Hamiltonian is a particular case of the super-Hamiltonian given in [18], up to the sign of V_{ij} and an additive scalar constant.

Because the Young tableaux for B and A belong to the class (38) and can differ by no more than the position of one cell (see [30], chapter 7, section 13), B can either coincide with A or have the form $(N-n\mp 1, 1^{n\pm 1})$.

Let us consider the case $B = (N-n-1, 1^{n+1})$, for which we can realize (see [18]) the Clebsch–Gordan coefficients in the operators $\mathbf{D}^A$ as

$$(\xi\alpha|\sigma) = \langle\alpha_n\ldots\alpha_1\xi|\sigma_1\ldots\sigma_{n+1}\rangle = \langle\sigma_{n+1}\ldots\sigma_1|\xi\alpha_1\ldots\alpha_n\rangle. \tag{47}$$

One can check that the Clebsch–Gordan coefficients, defined by equation (47), satisfy equation (24), and therefore they correctly connect the representations $A = (N-n, 1^n)$, $\Gamma = (N-1, 1)$ and $B = (N-n-1, 1^{n+1})$. These Clebsch–Gordan coefficients may differ from the standard ones (see [30]) by an inessential overall factor.

Now we can express the intertwining operators $\mathbf{D}^A$ (25) in terms of the fermionic operators defined above:

$$\begin{aligned}
\mathbf{D}^A\mathbf{e}_\beta &= \mathbf{e}_\sigma(\mathbf{e}_\sigma)^\dagger\mathbf{D}^A\mathbf{e}_\beta = \mathbf{e}_\sigma D^A_{\sigma\beta}\\
&= |\sigma_1\ldots\sigma_{n+1}\rangle\langle\sigma_{n+1}\ldots\sigma_1|\xi\alpha_1\ldots\alpha_n\rangle R_{\xi k}\left[-\mathrm{i}\partial_k\delta_{\beta\alpha} + \mathrm{i}\sum_{m\neq k}V_{km}\left(T^A_{km}\right)_{\beta\alpha}\right]\\
&= R_{\xi k}\left[-\mathrm{i}\partial_k|\xi\beta_1\ldots\beta_n\rangle + \mathrm{i}\sum_{m\neq k}V_{km}\left(T^A_{km}\right)_{\beta\alpha}|\xi\alpha_1\ldots\alpha_n\rangle\right]\\
&= R_{\xi k}\phi^+_\xi\left[-\mathrm{i}\partial_k|\beta_1\ldots\beta_n\rangle + \mathrm{i}\sum_{m\neq k}V_{km}\left(T^A_{km}\right)_{\beta\alpha}|\alpha_1\ldots\alpha_n\rangle\right]\\
&= R_{\xi k}\phi^+_\xi\left[-\mathrm{i}\partial_k + \mathrm{i}\sum_{m\neq k}V_{km}K_{km}\right]|\beta_1\ldots\beta_n\rangle.
\end{aligned}$$

So, we can conclude that

$$\mathbf{D}^A = R_{\xi k}\phi_\xi^+\left[-\mathrm{i}\partial_k + \mathrm{i}\sum_{m\neq k} V_{km}K_{km}\right]. \tag{48}$$

The operator $\mathbf{D}^A$ turns out not to depend on the specific choice of A from the class (38), i.e. on the fermionic number. It augments the fermionic number by one; this is related to the fact that $\mathbf{D}^A$ changes the position of one cell in the Young tableau for A (see [18]).

One can notice that

$$R_{\xi k}\phi_\xi^+ = \psi_k^+ - \frac{1}{N}\sum_m \psi_m^+ \tag{49}$$

and after some algebra one can deduce from (48):

$$\mathbf{D}^A = \mathrm{i}\phi_N^+\frac{\partial}{\partial y_N} + \psi_k^+\left[-\mathrm{i}\partial_k + \mathrm{i}\sum_{m\neq k} V_{km}K_{km}\right]. \tag{50}$$

To further simplify the form of the intertwining operator $\mathbf{D}^A$, one can use the following.

Theorem 2. *For all V_{km} with $V_{km} = -V_{mk}$ the following equation is satisfied*

$$\sum_{m\neq k}\psi_l^+ V_{km}K_{km} = \sum_{m\neq k}\psi_l^+ V_{km}$$

where K_{km} is the fermionic permutation operator (44).

The proof of the theorem 2 can be found in appendix B.

Now one can rewrite equation (50) as

$$\mathbf{D}^A = -\mathrm{i}q^+ \qquad q^+ \equiv -\phi_N^+\frac{\partial}{\partial y_N} + \psi_k^+\left[\partial_k - \sum_{m\neq k} V_{km}\right]. \tag{51}$$

Therefore, equation (28) with the Hamiltonian (46) takes the form

$$[H, q^+] = 0 \tag{52}$$

and its Hermitian conjugation[19] is

$$[H, q^-] = 0 \qquad q^- \equiv (q^+)^\dagger. \tag{53}$$

The operators $q^\pm$ (51) coincide with the supercharge operators[20] $q^\pm$ for the Calogero-like models given in [18], except for the sign of V_{ij}, which is determined by the sign of the constant l in equations (4)–(9). Therefore, if we replace l by $-l$ in the SUSY QM relations of [18], they can be rewritten as

$$\{q^-, q^+\} = H + \frac{\partial^2}{\partial y_N^2} + C \qquad (q^+)^2 = (q^-)^2 = 0. \tag{54}$$

The term $\frac{\partial^2}{\partial y_N^2}$ in equation (54) is unimportant because it commutes with $q^\pm$ and H.

The commutation relations (52) and (53) may be considered as the SUSY QM commutation relations, corresponding to the algebra (54) with the supercharges $q^\pm$ and the super-Hamiltonian $H + \frac{\partial^2}{\partial y_N^2} + C$.

[19] Equation (53) could also be obtained in another way; we could consider equation (28) with $A = (N-n, 1^n)$ but with $B = (N-n+1, 1^{n-1})$. Using formulae similar to equations (47)–(51) one can then check that $\mathbf{D}^A = \mathrm{i}q^- = \mathrm{i}(q^+)^\dagger$.

[20] The operator q^+ from equation (51) differs from the standard supercharge operator for the TCS model [17] by the term $-\phi_N\frac{\partial}{\partial y_N}$ that cancels the dependence of the supercharge on the centre-of-mass coordinates y_N, ϕ_N.

We can conclude that, for the models without OT, the intertwining relations (28) for the representations A and B from the class (38) turn into the relations of SUSY QM [18, 23]. For other A and B, equation (28) can be considered as a generalization of the SUSY QM intertwining relations[21].

Now let us turn to the CO model and the intertwining relations (29) with A from the class (38). From equations (45) and (18) it follows that the Hamiltonians $\mathbf{H}^A$ and $\mathbf{H}^B$ in equation (28) are

$$H = -\Delta + \omega^2 \sum_i x_i^2 + \sum_{i\neq j} \frac{l(l-K_{ij})}{(x_i - x_j)^2} + N\omega. \tag{55}$$

This Hamiltonian, analogously to the previous case (46), has the same form for all representations A from the class (38). However, the Hamiltonian (55) differs slightly from the corresponding Calogero Hamiltonian given in [18], as explained below.

The intertwining operators $\mathbf{D}^{A\pm}$ can be treated similarly to the case without OT, the only difference being that one should write $\partial_i \mp \omega x_i$ instead of ∂_i everywhere. In particular, for the case with $B = (N-n-1, 1^{n+1})$ the definition (30) leads to the following analogue of the formula (48) for $\mathbf{D}^{A\pm}$ (the same for all A from the class (38)):

$$\begin{aligned}\mathbf{D}^{A\pm} &= R_{\xi k}\phi_\xi^+\left[-\mathrm{i}\partial_k \pm \mathrm{i}\omega x_k + \mathrm{i}\sum_{m\neq k}(x_k - x_m)^{-1}K_{km}\right]\\ &= R_{\xi k}\phi_\xi^+\left[-\mathrm{i}\partial_k + \mathrm{i}\sum_{m\neq k}\left(\pm\frac{\omega}{N}(x_k - x_m) + (x_k - x_m)^{-1}K_{km}\right)\right].\end{aligned} \tag{56}$$

Making use of equation (49) and of theorem 2, similarly to equation (51) we can obtain that

$$\begin{aligned}
&\mathbf{D}^{A+} = -\mathrm{i}q^+ && q^+ \equiv -\phi_N^+\frac{\partial}{\partial y_N} + \psi_k^+\left[\partial_k - \sum_{m\neq k} W_{km}\right]\\
&W_{km} = W(x_k - x_m) && W(x) \equiv \frac{\omega}{N}x + \frac{l}{x}\\
&\mathbf{D}^{A-} = -\mathrm{i}\tilde{q}^+ && \tilde{q}^+ \equiv -\phi_N^+\frac{\partial}{\partial y_N} + \psi_k^+\left[\partial_k - \sum_{m\neq k} \tilde{W}_{km}\right]\\
&\tilde{W}_{km} = \tilde{W}(x_k - x_m) && \tilde{W}(x) \equiv -\frac{\omega}{N}x + \frac{l}{x}.
\end{aligned} \tag{57}$$

Taking into account the formulae (57) and (55), we can rewrite equation (29) for $\mathbf{D}^{A+}$ and A from the class (38) as

$$[H, q^+] = 2\omega q^+ \qquad [H, \tilde{q}^+] = -2\omega\tilde{q}^+ \tag{58}$$

and its Hermitian conjugation[22]

$$[H, q^-] = -2\omega q^- \qquad [H, \tilde{q}^-] = 2\omega\tilde{q}^- \tag{59}$$

[21] The intertwining relations are the most important part of the SUSY QM algebra, which is clear from a number of generalizations of the standard SUSY QM; see, for example, [25, 26].

[22] Equations (59) could also be obtained in another way; we could consider equation (29) with $A = (N-n, 1^n)$ but with $B = (N-n+1, 1^{n-1})$. Using formulae similar to (47)–(51), (56) and (57) one can then check that $\mathbf{D}^{A+} = \mathrm{i}\tilde{q}^-$, where q^- is defined in equation (60), and $\mathbf{D}^{A-} = \mathrm{i}q^-$.

$$
\begin{aligned}
q^- &= (q^+)^\dagger = \phi_N \frac{\partial}{\partial y_N} + \psi_k \left[-\partial_k - \sum_{m\neq k} W_{km} \right] \\
\tilde{q}^- &= (\tilde{q}^+)^\dagger = \phi_N \frac{\partial}{\partial y_N} + \psi_k \left[-\partial_k - \sum_{m\neq k} \tilde{W}_{km} \right].
\end{aligned}
\tag{60}
$$

The operators $q^\pm$ from equations (57) and (60) are similar to the supercharge operators[23] $q^\pm$ given in [18] for the Calogero model with OT, up to a redefinition of constants. Therefore, we can construct the following SUSY algebra

$$
\{q^-, q^+\} = h \qquad (q^+)^2 = (q^-)^2 = 0 \tag{61}
$$

with the super-Hamiltonian h

$$
h = H + 2\omega\psi_k^+\psi_k - H_N \qquad H_N = -\frac{\partial^2}{\partial y_N^2} + \omega^2 y_N^2 + 2\omega\phi_N^+\phi_N + C \tag{62}
$$

where H is defined in equation (55), and C is a scalar constant. The term H_N is unimportant because it commutes with $q^\pm$ and H. The operators $\tilde{q}^\pm$ form an algebra similar to equation (61) but the sign of ω in the super-Hamiltonian (62) should be different (see [36]):

$$
\begin{aligned}
&\{\tilde{q}^-, \tilde{q}^+\} = \tilde{h} && (\tilde{q}^+)^2 = (\tilde{q}^-)^2 = 0 \\
&\tilde{h} = H - 2\omega\psi_k^+\psi_k - \tilde{H}_N && \tilde{H}_N = -\frac{\partial^2}{\partial y_N^2} + \omega^2 y_N^2 - 2\omega\phi_N^+\phi_N + \tilde{C}.
\end{aligned}
$$

From the SUSY algebra (61) and (63) one can deduce the commutation relations that can be shown to be equivalent to equations (58) and (59):

$$
[h, q^\pm] = 0 \qquad [\tilde{h}, \tilde{q}^\pm] = 0.
$$

Equations (28) and (29) in the case of A and B from the class (38) are reduced to the ordinary multi-dimensional SUSY QM [23] for the Calogero-like models [15, 18]. However, for A or B outside that class, equations (28) and (29) describe a generalization of the SUSY QM intertwining relations that has not been known before. Clearly, the SUSY QM is valid not only for the Calogero-like models, but for many others [23]. The question is how far the generalization of SUSY QM constructed above can be extended to other, non Calogero-like models, and this deserves further attention.

Acknowledgments

This work has been made possible in part by the support provided by a grant from the Russian Foundation of Basic Research (No 02-01-00499). MI is indebted to DAAD (Germany) and to the University of Helsinki for their support. He also warmly acknowledges the kind hospitality and useful discussions with Professors H-J Korsch (Kaiserslautern) and M Chaichian (Helsinki).

[23] The operator q^+ from equation (57) differs from the standard supercharge operator for the TCS model [14] by the term $-\phi_N^+ \frac{\partial}{\partial y_N}$, which cancels the dependence of the supercharge on the centre-of-mass coordinates y_N and ϕ_N.

Appendix A

In this appendix, we prove the following.

Theorem 1. *Let A be some representation of S_N. Let $L_{\alpha\beta}$ be some linear differential operator of finite order with the coefficients being rational matrix functions of the variables x_i, or* $\sin x_i, \cos x_i$, *or* $\mathrm{sh}x_i, \mathrm{ch}x_i$ *(but no two of them simultaneously), singular at $U = \{\mathbf{x} | \exists i, j : i \neq j, x_i = x_j\}$ at most. The coefficients are matrices of dimension* $\dim A \times \dim A$. *Then, if*

$$L_{\alpha\beta} f_\beta = 0$$

for all f_β satisfying (13), then $L \equiv 0$ as an operator.

Proof. Consider the principal Veyl chamber: $\{\mathbf{x} : x_1 < \cdots < x_N\}$. Every function defined on this chamber can be continued on to the rest of $\Re^{\dim A}$ by using equation (13). The result will obviously satisfy equation (13), so it is annihilated by L. Hence, L annihilates all functions on the principal Veyl chamber. The same can be stated about every other Veyl chamber: $\{\mathbf{x} : x_{i_1} < \cdots < x_{i_N}\}$. Hence, L annihilates all functions on $\Re^{\dim A} \setminus U$. From the fact that the coefficients of L are rational functions of x_i, or $\sin x_i$, $\cos x_i$, or $\mathrm{sh}x_i$, $\mathrm{ch}x_i$, it then follows that they are zero identically. □

Appendix B

In this appendix, we prove the following.

Theorem 2. *For all V_{km} such that $V_{km} = -V_{mk}$*

$$\sum_{m\neq k} \psi_k^+ V_{km} K_{km} = \sum_{m\neq k} \psi_k^+ V_{km} \tag{63}$$

where K_{km} is the fermionic permutation operator defined in equation (44).

Proof. Taking into account the definition (44), we can check that

$$\psi_k^+ K_{km} = \psi_k^+ + \psi_k^+ \psi_m^+ \psi_m + \psi_m^+ \psi_k^+ \psi_k. \tag{64}$$

In equation (64) no summation over either index is implied. Substituting equation (64) into the left-hand side of equation (63) we see that

$$\begin{aligned}\sum_{m\neq k} \psi_k^+ V_{km} K_{km} &= \sum_{m\neq k} V_{km} \left[\psi_k^+ + \psi_k^+ \psi_m^+ \psi_m + \psi_m^+ \psi_k^+ \psi_k\right] \\ &= \sum_{m\neq k} \left[V_{km} \psi_k^+ + V_{km} \psi_k^+ \psi_m^+ \psi_m + V_{mk} \psi_k^+ \psi_m^+ \psi_m\right] = \sum_{m\neq k} \psi_k^+ V_{km}.\end{aligned}$$

□

References

[1] Calogero F 1969 *J. Math. Phys.* **10** 2191
Calogero F 1969 *J. Math. Phys.* **10** 2197
[2] Calogero F 1971 *J. Math. Phys.* **12** 419
[3] Sutherland B 1971 *Phys. Rev.* A **4** 2019
[4] Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 6
[5] Rühl W and Turbiner A 1995 *Mod. Phys. Lett.* A **10** 2213
[6] Minahan J A and Polychronakos A P 1993 *Phys. Lett.* B **302** 265
[7] Dodlov O V, Konstein S E and Vasiliev M A 1993 *Preprint* hep-th/9311028
[8] Pasquier V 1994 *Preprint* hep-th/9405104

[9] Cannata F and Ioffe M V 2001 *J. Phys. A: Math. Gen.* **34** 1129
[10] Calogero F and Marchioro C 1973 *J. Math. Phys.* **14** 182
Khare A and Ray K 1997 *Phys. Lett.* A **230** 139
[11] Ghosh P, Khare A and Sivakumar M 1998 *Phys. Rev.* A **58** 821
[12] Khastgir S P, Pocklington A J and Sasaki R 2000 *J. Phys. A: Math. Gen.* **33** 9033
[13] Ghosh P K 1997 *Phys. Lett.* A **229** 203
[14] Freedman D Z and Mende P F 1990 *Nucl. Phys.* B **344** 317
[15] Efthimiou C and Spector H 1997 *Phys. Rev.* A **56** 208
[16] Brink L, Turbiner A and Wyllard N 1998 *J. Math. Phys.* **39** 1285
Shastry B S and Sutherland B 1993 *Phys. Rev. Lett.* **70** 4029
[17] Desrosiers P, Lapointe L and Mathieu P 2001 *Nucl. Phys.* B **606** 547
[18] Ioffe M V and Neelov A I 2000 *J. Phys. A: Math. Gen.* **33** 1581
[19] Dunkl C F 1989 *Trans. Am. Math. Soc.* **311** 167
[20] Polychronakos A P 1992 *Phys. Rev. Lett.* **69** 703
[21] Brink L, Hansson T H and Vasiliev M A 1992 *Phys. Lett.* B **286** 109
[22] Brink L, Hansson T H, Konstein S E and Vasiliev M A 1993 *Nucl. Phys.* B **401** 591
[23] Andrianov A A, Borisov N V, Ioffe M V and Eides M I 1984 *Phys. Lett. A: Math. Gen.* **109** 143
Andrianov A A, Borisov N V, Ioffe M V and Eides M I 1985 *Theor. Math. Phys.* **61** 965 (Transl. from 1984 *Teor. Mat. Fiz.* **61** 17)
Andrianov A A, Borisov N V and Ioffe M V 1984 *Phys. Lett.* A **105** 19
Andrianov A A, Borisov N V and Ioffe M V 1985 *Theor. Math. Phys.* **61** 1078 (Transl. from 1984 *Teor. Mat. Fiz.* **61** 183)
[24] Bordner A J, Manton N S and Sasaki R 2000 *Prog. Theor. Phys.* **103** 463
[25] Andrianov A A, Ioffe M V, Cannata F and Dedonder J-P 1995 *Int. J. Mod. Phys.* A **10** 2683
Andrianov A A, Ioffe M V and Nishnianidze D N 1999 *J. Phys. A: Math. Gen.* **32** 4641
[26] Cannata F, Ioffe M V and Nishnianidze D N 2002 *J. Phys. A: Math. Gen.* **35** 1389
[27] Witten E 1981 *Nucl. Phys.* B **188** 513
Witten E 1982 *Nucl. Phys.* B **202** 253
Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **25** 268
[28] Gendenstein L E 1983 *JETP. Lett.* **38** 356
[29] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics* vol III (New York: Academic)
[30] Hamermesh M 1964 *Group Theory and its Application to Physical Problems* (New York: Addison-Wesley)
[31] Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)
[32] Sasaki R and Takasaki K 2001 *J. Phys. A: Math. Gen.* **34** 9533
[33] Aoyama H, Sato M and Tanaka T 2001 *Phys. Lett.* B **503** 423
[34] Bohr A and Mottelson B R 1969 *Nuclear Structure* (New York: Benjamin)
[35] Berestetskii V B, Lifshitz E M and Pitayevskii L M 1982 *Quantum Electrodynamics* (Oxford: Pergamon)
[36] Ghosh P K 2001 *Nucl. Phys.* B **595** 519